STA 360/602L: Module 1.1

Building blocks of Bayesian inference

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Building blocks of Bayesian inference

- Generally (unless otherwise stated), in this course, we will use the following notation. Let
  - $\mathcal{Y}$ be the sample space;
  - y be the observed data;
  - Θ be the parameter space; and
  - θ be the parameter of interest.
- More to come later.

Frequentist inference

- Given data y, estimate the population parameter θ.
- How to estimate θ under the frequentist paradigm?
  - Maximum likelihood estimate (MLE)
  - Method of moments
  - and so on...
- Frequentist ML estimation finds the one value of θ that maximizes the likelihood.
- Typically uses large sample (asymptotic) theory to obtain confidence intervals and do hypothesis testing.

What are Bayesian methods?

- Bayesian methods are data analysis tools derived from the principles of Bayesian inference and provide
  - parameter estimates with good statistical properties;
  - parsimonious descriptions of observed data;
  - predictions for missing data and forecasts of future data; and
  - a computational framework for model estimation, selection, and validation.

Bayes' theorem - basic conditional probability

- Let's take a step back and quickly review the basic form of Bayes' theorem.
- Suppose there are some events A and B having probabilities $\Pr(A)$ and $\Pr(B)$.
- Bayes' rule gives the relationship between the marginal probabilities of A and B and the conditional probabilities.
- In particular, the basic form of Bayes' rule or Bayes' theorem is

$$\Pr(A|B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

$\Pr(A)$ = marginal probability of event A, $\Pr(B|A)$ = conditional probability of event B given event A, and so on.

Building blocks of Bayesian inference

- Now, to a slightly more complicated version of Bayes' rule. First,

  1. For each $\theta \in \Theta$, specify a prior distribution $p(\theta)$ or $\pi(\theta)$, describing our beliefs about θ being the true population parameter.

  2. For each $\theta \in \Theta$ and $y \in \mathcal{Y}$, specify a sampling distribution $p(y|\theta)$, describing our belief that the data we see y is the outcome of a study with true parameter θ. $p(y|\theta)$ gets us the likelihood $L(\theta|y)$.

  3. After observing the data y, for each $\theta \in \Theta$, update the prior distribution to a posterior distribution $p(\theta|y)$ or $\pi(\theta|y)$, describing our "updated" belief about θ being the true population parameter.

- Now, how do we get from Step 1 to 3? Bayes' rule!

$$p(\theta|y) = \frac{p(\theta)p(y|\theta)}{\int_\Theta p(\tilde{\theta})p(y|\tilde{\theta})\mathrm{d}\tilde{\theta}} = \frac{p(\theta)p(y|\theta)}{p(y)}$$

We will use this over and over throughout the course!

Notes on prior distributions

Many types of priors may be of interest. These may

- represent our own beliefs;
- represent beliefs of a variety of people with differing prior opinions; or
- assign probability more or less evenly over a large region of the parameter space.
- and so on...

Notes on prior distributions

- Subjective Bayes: a prior should accurately quantify some individual's beliefs about θ.
- Objective Bayes: the prior should be chosen to produce a procedure with "good" operating characteristics without including subjective prior knowledge.
- Weakly informative: prior centered in a plausible region but not overly-informative, as there is a tendency to be over confident about one's beliefs.

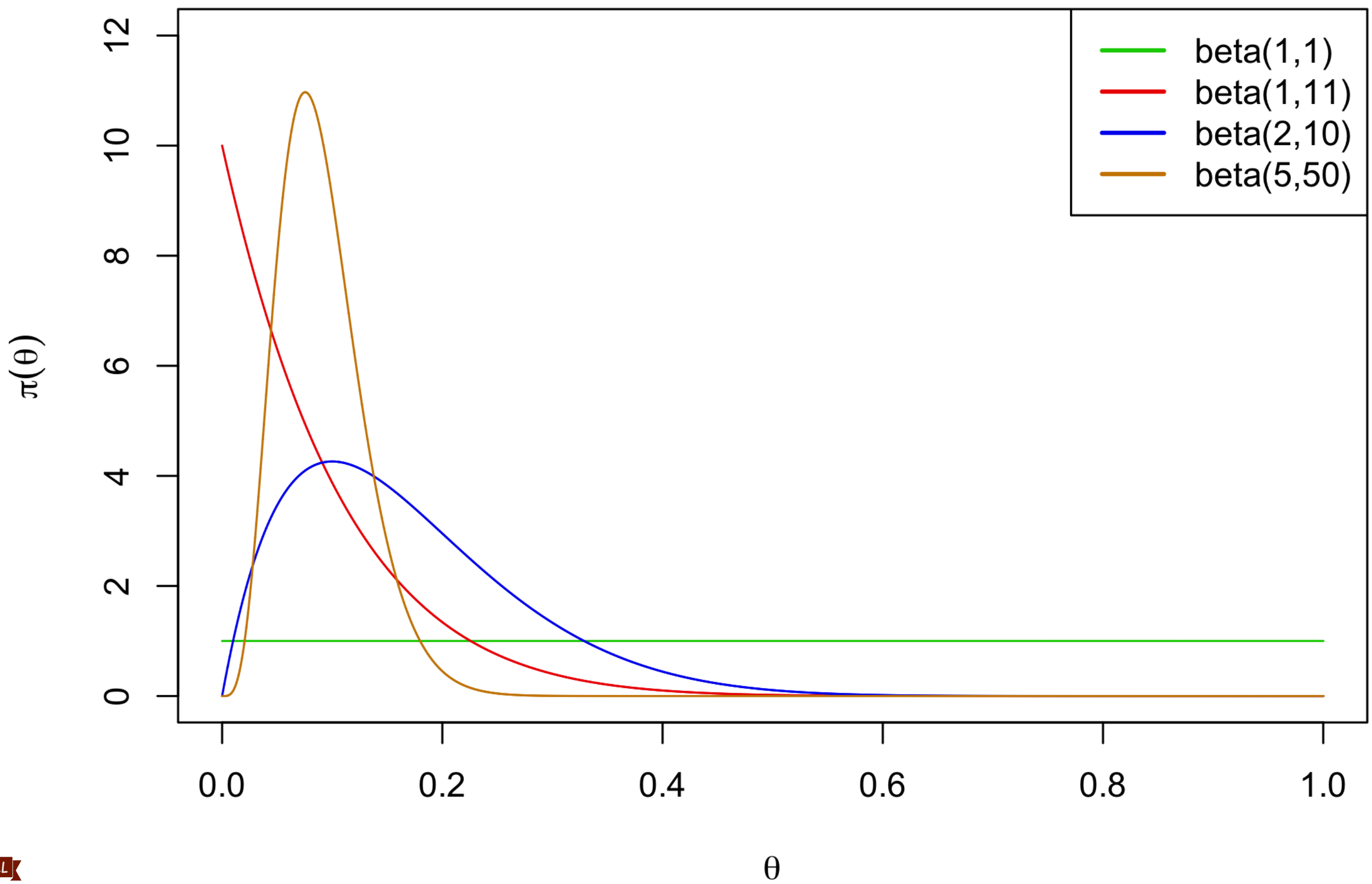
Notes on prior distributions

- The prior quantifies your initial uncertainty in θ before you observe new data (new information) - this may be necessarily subjective & summarize experience in a field or prior research.
- Even if the prior is not "perfect", placing higher probability in a ballpark of the truth leads to better performance.
- Hence, it is very seldom the case that a weakly informative prior is not preferred over no prior.
- One (very important) role of the prior is to stabilize estimates in the presence of limited data.

Simple example - estimating a population proportion

- Suppose $\theta \in (0, 1)$ is the population proportion of individuals with diabetes in the US.
- A prior distribution for θ would correspond to some distribution that distributes probability across $(0, 1)$.
- A very precise prior corresponding to abundant prior knowledge would be concentrated tightly in a small sub-interval of $(0, 1)$.
- A vague prior may be distributed widely across $(0, 1)$ - e.g., a uniform distribution would be the common choice here.

Some possible prior densities



Beta prior densities

- These four priors correspond to $\mathrm{Beta}(1,1)$ (also $\mathrm{Unif}(0,1)$), $\mathrm{Beta}(1,10)$, $\mathrm{Beta}(2,10)$ and $\mathrm{Beta}(5,50)$ densities.

- Beta(a,b) is a probability density function (pdf) on (0,1),

$$\pi(\theta) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1},$$

  where $B(a,b)$ = beta function = normalizing constant ensuring the kernel integrates to one. Note: some texts write $\mathrm{beta}(\alpha,\beta)$ instead.

- The beta(a,b) distribution has expectation $\mathbb{E}[\theta] = a/(a+b)$ and the density becomes more and more concentrated as $a+b$ = prior "sample size" increases.

- The variance $\mathbb{V}[\theta] = ab/[(a+b)^2(a+b+1)]$.

- We will look more carefully into the beta-binomial model soon but first, we will explore how this prior gets updated as data becomes available, during the online discussion session.

What's next?

Move on to the readings for the next module!