STA 360/602L: Module 2.7

Gamma-Poisson model I

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Poisson distribution recap

- $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Poisson}(\theta)$ denotes that each Y_i is a Poisson random variable.

- The Poisson distribution is commonly used to model count data consisting of the number of events in a given time interval.

- Some examples: # children, # lifetime romantic partners, # songs on iPhone, # tumors on mouse, etc.

- The Poisson distribution is parameterized by θ and the pmf is given by

$$\Pr[Y_i = y_i | \theta] = \frac{\theta^{y_i} e^{-\theta}}{y_i!}; \quad y_i = 0, 1, 2, \ldots; \quad \theta > 0.$$

  where

$$\mathbb{E}[Y_i] = \mathbb{V}[Y_i] = \theta.$$

- What is the joint likelihood? What is the best guess (MLE) for the Poisson parameter? What is the sufficient statistic for the Poisson parameter?

Gamma density recap

- The gamma density will be useful as a prior for parameters that are strictly positive.
- If $\theta \sim \text{Ga}(a, b)$, we have the pdf

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

  where a is known as the shape parameter and b, the rate parameter.
- Another parameterization uses the scale parameter $\phi = 1/b$ instead of b.
- Some properties:
  - $\mathbb{E}[\theta] = \dfrac{a}{b}$
  - $\mathbb{V}[\theta] = \dfrac{a}{b^2}$
  - $\text{Mode}[\theta] = \dfrac{a-1}{b} \quad \text{for} \quad a \geq 1$

Gamma density

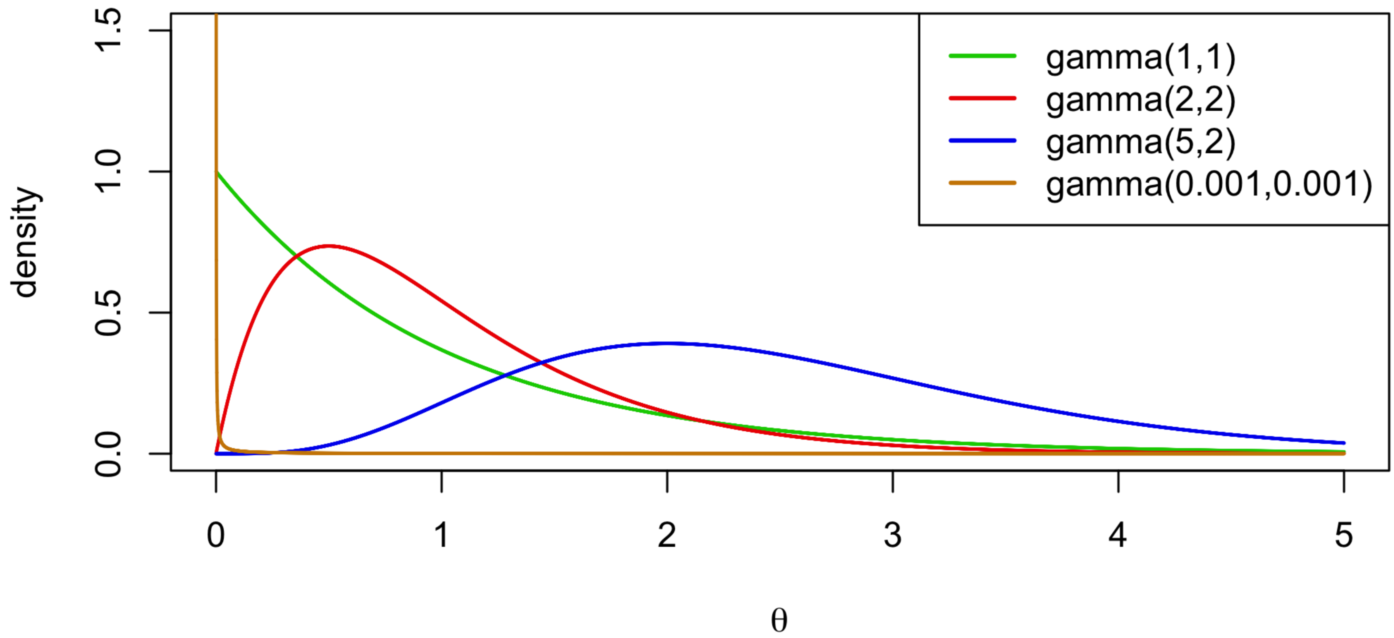
- If our prior guess of the expected count is μ & we have a prior "scale" ϕ, we can let

$$\mathbb{E}[\theta] = \mu = \frac{a}{b}; \;\; \mathbb{V}[\theta] = \mu\phi = \frac{a}{b^2},$$

  and solve for a, b. We can play the same game if we have a prior variance or standard deviation.

- More properties:
  - If $\theta_1, \ldots, \theta_p \overset{ind}{\sim} \text{Ga}(a_i, b)$, then $\sum_i \theta_i \sim \text{Ga}(\sum_i a_i, b)$.
  - If $\theta \sim \text{Ga}(a, b)$, then for any $c > 0$, $c\theta \sim \text{Ga}(a, b/c)$.
  - If $\theta \sim \text{Ga}(a, b)$, then $1/\theta$ has an Inverse-Gamma distribution. *We'll take advantage of these soon!*

Example gamma distributions



R has the option to specify either the rate or scale parameter so always make sure to specify correctly when using "dgamma","rgamma",etc!.

Gamma-Poisson

Generally, it turns out that

Poisson data:

$$p(y_i|\theta) : y_1, \ldots, y_n \overset{iid}{\sim} \text{Poisson}(\theta)$$

$+$ Gamma Prior:

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} = \text{Ga}(a, b)$$

$\Rightarrow$ Gamma posterior:

$$\pi(\theta|\{y_i\}) : \theta|\{y_i\} \sim \text{Ga}(a + \sum y_i, b + n).$$

That is, updating a gamma prior with a Poisson likelihood leads to a gamma posterior – we once again have conjugacy.

Can we derive the posterior distribution and its parameters? Let's do some work on the board.

Gamma-Poisson

- With $\pi(\theta|\{y_i\}) = \text{Ga}(a + \sum y_i, b + n)$, we can think of
  - b as the "number prior of observations" from some past data, and
  - a as the "sum of the counts from the b prior observations".
- Using the properties of the gamma distribution, we have
  - $\mathbb{E}[\theta|\{y_i\}] = \dfrac{a + \sum y_i}{b + n}$
  - $\mathbb{V}[\theta|\{y_i\}] = \dfrac{a + \sum y_i}{(b + n)^2}$
- So, as we did with the beta-binomial, we can once again write the posterior expectation as a weighted average of prior and data.

$$\mathbb{E}(\theta|\{y_i\}) = \frac{a + \sum y_i}{b + n} = \frac{b}{b + n} \times \text{prior mean} + \frac{n}{b + n} \times \text{MLE}.$$

- Again, as we get more and more data, the majority of our information about θ comes from the data as opposed to the prior.

Hoff example: birth rates

- Survey data on educational attainment and number of children of 155 forty-year-old women during the 1990's.
- These women were in their 20s during the 1970s, a period of historically low fertility rates in the US.
- **Goal**: compare birth rate θ_1 for women with bachelor's degrees to the rate θ_2 for women without.
- **Data**:
  - 111 women without a bachelor's degree had 217 children: $(\bar{y}_1 = 1.95)$
  - 44 women with bachelor's degrees had 66 children: $(\bar{y}_2 = 1.50)$
- Based on the data alone, looks like θ_1 should be greater than θ_2. But...how sure are we?
- **Priors**: $\theta_1, \theta_2 \sim \text{Ga}(2, 1)$ (not much prior information; equivalent to 1 prior woman with 2 children). Posterior means will be close to the MLEs.

Hoff example: birth rates

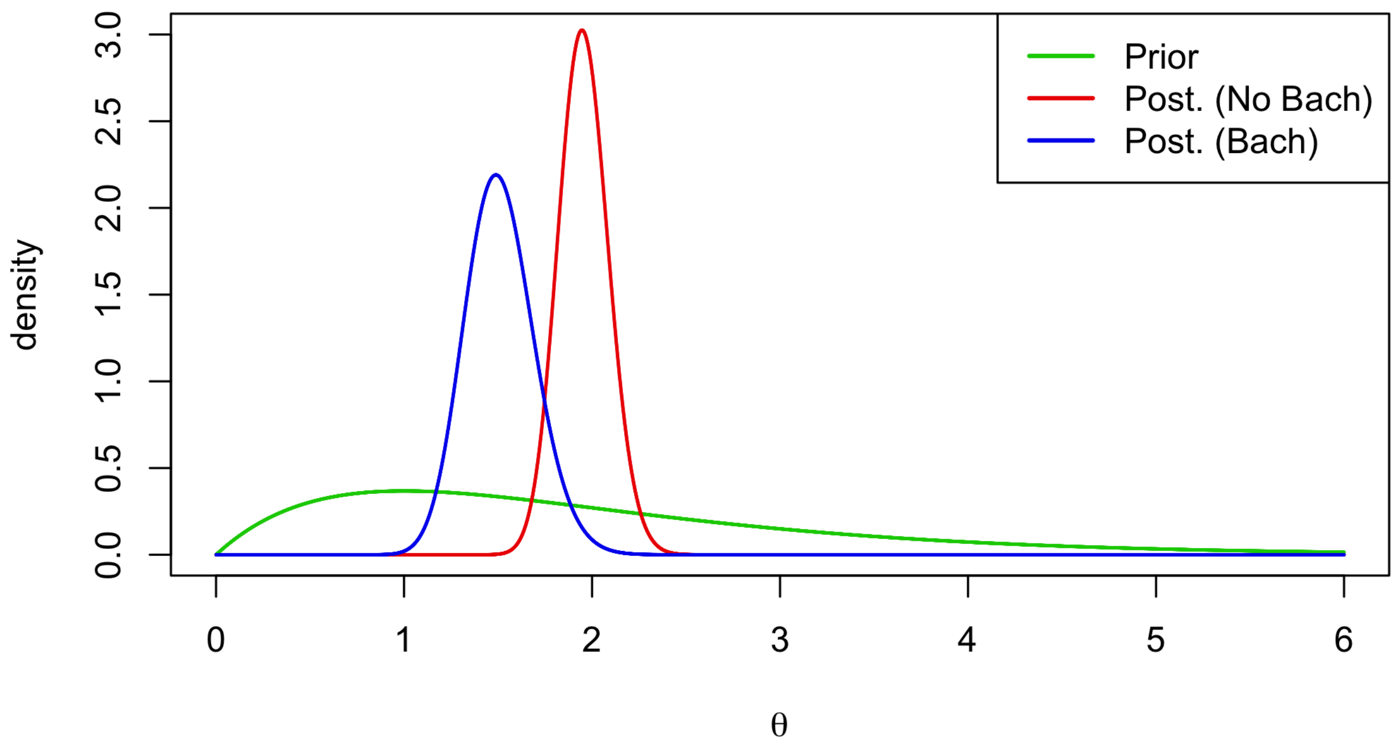
- Then,
  - $\theta_1|\{n_1 = 111, \sum y_{i,1} = 217\} \sim \text{Ga}(2 + 217, 1 + 111) = \text{Ga}(219, 112)$.
  - $\theta_2|\{n_2 = 44, \sum y_{i,2} = 66\} \sim \text{Ga}(2 + 66, 1 + 44) = \text{Ga}(68, 45)$.
- Use R to calculate posterior means and 95% CIs for θ_1 and θ_2.

```
a=2; b=1; #prior
n1=111; sumy1=217; n2=44; sumy2=66 #data
(a+sumy1)/(b+n1); (a+sumy2)/(b+n2); #post means
qgamma(c(0.025, 0.975),a+sumy1,b+n1) #95\% ci 1
qgamma(c(0.025, 0.975),a+sumy2,b+n2) #95\% ci 2
```

- Posterior means: $\mathbb{E}[\theta_1|\{y_{i,1}\}] = 1.955$ and $\mathbb{E}[\theta_2|\{y_{i,2}\}] = 1.511$.
- 95% credible intervals
  - θ_1: [1.71, 2.22].
  - θ_2: [1.17, 1.89].

Hoff example: birth rates

Prior and posteriors:



Hoff example: birth rates

- Posteriors indicate considerable evidence birth rates are higher among women without bachelor's degrees.
- Confirms what we observed.
- Using sampling we can quickly calculate $\Pr(\theta_1 > \theta_2 | \text{data})$.

```
mean(rgamma(10000,219,112)>rgamma(10000,68,45))
```

  We have $\Pr(\theta_1 > \theta_2 | \text{data}) = 0.97$.

- Why/how does it work?
- Monte Carlo approximation coming soon!
- Clearly, that probability will change with different priors.

What's next?

Move on to the readings for the next module!